THE OPTIMAL SYMMETRICAL POINTS FOR POLYNOMIAL INTERPOLATION OF REAL FUNCTIONS IN THE TETRAHEDRON

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20. ABSTRACT

The main result of this paper is the computation of the mean optimal symmetrical interpolation points in the tetrahedron up to degree 9. This interpoation set has the smallest Lebesgue constant known today.

The Optimal Symmetrical Points for Polynomial Interpolation of Real Functions in the Tetrahedron

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Abstract

The main result of this paper is the computation of the mean optimal symmetrical interpolation points in the tetrahedron up to degree 9. This interpoation set has the smallest Lebesgue constant known today.

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High quality polynomial interpolation of functions is essential in many applications [1] [7], in particular, in the p-version of the Finite Element Method. However, very little is known about the approximation accuracy of interpolation in the simplex in several dimensions. Of particular concern are the two dimensional and three dimensional cases. We refer here to the few literatures that discuss this problem [2] [3] [4] [6].

In this paper, we compute the mean optimal interpolation sets in the tetrahedron. The main result in [3] is that in the interval and in the triangle, the mean optimal sets are close to the optimal interpolation sets in the mean norm(the smallest Lebesgue constant sets). We give the mean optimal sets in table 3 in this paper. In the literature, the most widely used interpolation sets in the tetrahedron is the equally spaced point sets. The approximation quality for equally spaced point sets is known to deteriorate considerably for high degree interpolation. Bos [2] has proposed a set for arbitrary dimension up to degree 4. Aside from these, We are unaware of any other high degree interpolation sets in the tetrahedron.

Interpolation in a simplex of dimensions more than three is still of interest even though it is not considered in this paper. The general principle introduced in [3] can still be employed, albeit in a more complicated fashion.

We use the same notation as is used in [3]. Let S^3 be the three dimensional simplex in R^3. Let $P_n(S^3)$ be the space of polynomials of degree n in three variables. Then dim $P_n(S^3) = N_n := \begin{pmatrix} n+3 \\ 3 \end{pmatrix} = \frac{(n+1)(n+2)(n+3)}{6}$. Let $T^n = (x_1, \cdots, x_{N_n})$, $x_j \in S^3$ be a set of distinct points in S^3, where n is the degree of interpolation. T^n will be called the nodal set and x_i the nodal points. Furthermore let $\mathcal{T}^n$ be the family of all nodal sets, or the set of admissible nodes (see next section for discussion of admissible nodes) *i.e.*, $T^n \in \mathcal{T}^n$. The interpolation problem now reads: Given a continuous function in S^3, $f \in C(S^3)$ and the nodal set T^n, find $p_n \in P_n$ such that $p_n(x_i) = f(x_i)$, $x_i \in T^n(S^3), i = 1, \cdots, N_n$. Let $m_j(x), j = 1, \cdots, N_n$ be the basis functions of $P_n(S^3)$, then $p_n(x_i) = \sum_{j=1}^{N_n} a_j m_j(x_i) = f(x_i), 1 \le i \le N_n$. This system

of linear equations is uniquely solvable for all right hand sides if and only if the determinant is nonzero. The determinant of the system is denoted by $VDM(T^n)$. Assuming now that $VDM(T^n) \neq 0$, we can construct the Lagrange coefficients $L_j^T(x) \in P_n(S^3), j = 1, ..N_n$ as basis of $P_n(S^3)$ with $L_j^T(x_i) = \delta_{i,j}, 1 \leq i, j \leq N_n$. For $f \in C(S^3)$, we have

$$p_n(x) = \sum_{k=1}^{N_n} f(x_k) L_k^T(x). \tag{1}$$

Denote $p_n(t) = \mathcal{L}_T f$, then $\mathcal{L}_T$ is a linear projection operator which maps $C(S^3)$ onto $P_n(S^3)$. If no misunderstanding will occur, we sometimes omit the index n.

We equip $P_n(S^3)$ with a norm $\|\cdot\|$ and denote

$$\|\mathcal{L}_T\| = \sup_{f \neq 0} \frac{\|\mathcal{L}_T f\|}{\|f\|_\infty}, \tag{2}$$

if $\|\cdot\| = \|\cdot\|_\infty$, we write $\|\mathcal{L}_T\|_\infty = \lambda(T)$. It is easy to show that:

$$\lambda(T) = \max_{x \in S^3} \sum_{k=1}^{N_n} |L_k^T(x)|. \tag{3}$$

$\lambda(T)$ is called the Lebesgue constant of $\mathcal{L}_T$. In addition, we define

$$\|\langle \mathcal{L}_T \rangle\|_2^2 := \int_{S^3} \sum_{k=1}^{N_n} |L_k^T(x)|^2 dx, \tag{4}$$

Now we are interested in the sets T_1 which minimizes $\|\mathcal{L}_T\|_\infty$ and T_2 which minimizes $\|\langle \mathcal{L}_T \rangle\|_2$. T_1 is the optimal interpolation set. T_2 is called the mean L^2 (or briefly mean) optimal set. We are also interested in the set T_{VDM} which maximizes $|VDM|$ and call it the VDM set. The VDM set is sometimes used to approximate the optimal set, see [2].

Remark: Let $f \in C(S^3)$, then $\|f - \mathcal{L}_T f\|_\infty \leq (1 + \lambda(T)) \inf_{g \in P_n(S^3)} \|f - g\|_\infty$, see [3]. Hence, the interpolation error is of the same order as the best approximation up to the Lebesgue constant. Interpolation points with the smallest Lebesgue constant therefore leads to the smallest interpolation error.

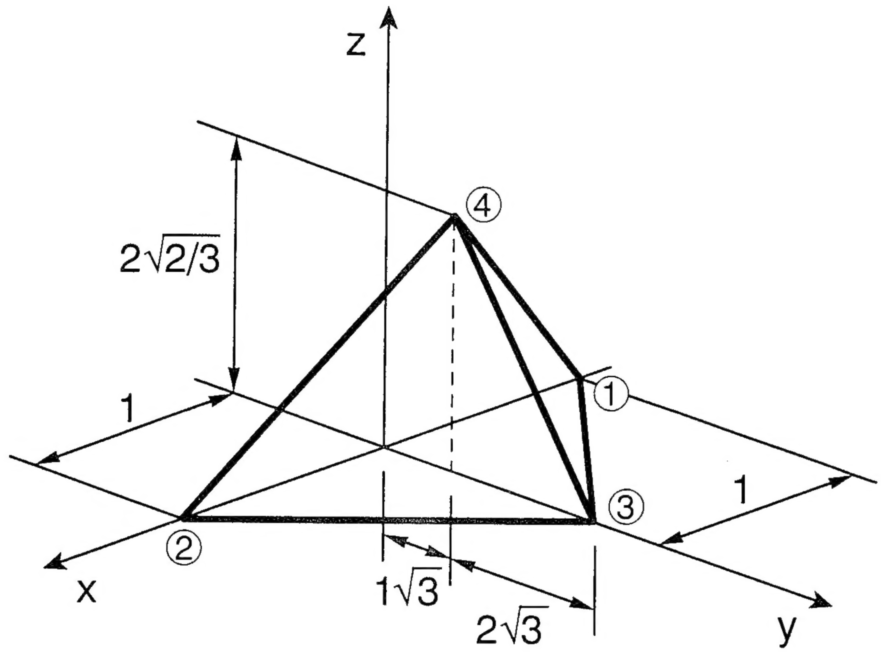

Figure 1: The standard tetrahedron.

From now on, S^3 is the standard tetrahedron shown in Fig. 1. We denote (x, y, z) the Cartesian coordinates.

We shall be interested in the degree n optimal interpolation set for the mean norm(or the VDM determinant) in a restricted set $\mathcal{T}$ of the nodal points:

a. The set of nodal points is symmetrical with respect to all symmetries of the tetrahedron;

b. On each face of the tetrahedron, we use degree n optimal interpolation set for the mean norm(or the VDM determinant) in the triangle.

Condition b. is based on the following consideration. When we approximate a function in a domain partitioned into tetrahedrons, we require the resulting piecewise polynomial to be continuous. We also want to have minimal error on each face. Condition b. is not a serious impediment. Indeed in the triangle case, we computed the degree n optimal set $\tilde{T}^n_{2Tri}$ for the mean norm under the constraint that the nodal points on the edges of the triangle is the one dimensional degree n mean optimal set. The corresponding Lebesgue constant is quite close to that of the actual mean norm optimal set T^n_{2Tri}. This is shown in Table 1.

Because finding optimal points T_1 is much more difficult, we minimize $\|\langle\mathcal{L}_T\rangle\|_2$ and find

Table 1: Lebesgue constant and the mean norm for the triangle optimal mean set T^n_{2Tri}, and the constrained mean set $\tilde{T}^n_{2Tri}$. Note the square of the mean norm is normalized by the trangle area.

n	$\lambda(T^n_{2Tri})$	$\lambda(\tilde{T}^n_{2Tri})$	$\frac{1}{\sqrt{3}}\|\langle\mathcal{L}_{T^n_{2Tri}}\rangle\|_2^2$	$\frac{1}{\sqrt{3}}\|\langle\mathcal{L}_{\tilde{T}^n_{2Tri}}\rangle\|_2^2$
2	$1\frac{2}{3}$	$1\frac{2}{3}$	$\frac{19}{30}$	$\frac{19}{30}$
3	2.1114	2.1206	0.7404	0.7420
4	2.6920	2.6833	0.8196	0.8220
5	3.3010	3.2695	0.8837	0.8863
6	3.7910	3.7736	0.9408	0.9434
7	4.3908	4.3949	0.9964	0.9990
8	5.0893	5.1106	1.0553	1.0578
9	5.9181	5.9456	1.1227	1.1252
10	7.0850	7.1140	1.2050	1.2076
11	8.3383	8.3864	1.3110	1.3140
12	8.6928	8.7384	1.4533	1.4572
13	12.0464	12.1120	1.6508	1.6561

the optimal set T_2 in the constrained family. More precisely we find a set which is an approximation of a local minimum of $\|\langle\mathcal{L}_T\rangle\|_2$. We also compute the maximum of $|VDM|$ subject to the same constraints. We point out that the tetrahedron VDM set T^n_{VDM} restricted to a face of the tetrahedron is exactly the two dimensional triangle VDM set computed in [3]. Therefore, constraint b. is automatically satisfied by the VDM set. This can be seen by using the hierarchic shape functions in [7] as the basis functions $m_j(x)$. Hierarchic shape functions for the tetrahedron are organized into four categories: the nodal shape functions, the edge modes, the face modes and the interior modes. $|VDM|$ in the hierarchic shape function basis is the product of $|VDM|$ for the edge modes, $|VDM|$ for the face modes and $|VDM|$ for the interior modes. Therefore, maximizing $|VDM|$ for the tetrahedron is equivalent to maximizing $|VDM|$ for each edge, each face and the interior.

By symmetry, a nodal point inside the tetrahedron can be at the center of the tetrahedron, or on one of the four lines passing through a vertex and the center of the triangle face opposite to that vertex, or on one of the three lines connecting the midpoints of two

nonintersecting edges, or on one of the six symmetry planes but on none of the aforementioned lines, or on none of the symmetry planes (this node is located inside one of the twenty four subtetrahedrons bounded by the six symmetry planes). It corresponds to a point in a singlet, or a quartet, or a sextet, or a twelve fold symmetrical point, or a twenty four fold symmetrical point. We denote the number of these multiplets by $n_1 (n_1 = 0 \text{ or } 1)$, n_4, n_6, n_{12}, n_{24}. Since the total number of these multiplets equals the number of interpolating nodes inside the tetrahedron, $n_1 + 4n_4 + 6n_6 + 12n_{12} + 24n_{24} = \frac{(n-1)(n-2)(n-3)}{6}$.

The integer solution for the above equation is nonunique when $n \geq 6$. Different integer solutions correspond to different symmetry patterns of the nodal set. Each symmetry pattern has a minimum mean norm, we want to find the smallest mean norm (the global minimum) among these minima.

For the n-th degree equally spaced point set, the symmetry pattern will be denoted as $(n_1^{eq}, n_4^{eq}, n_6^{eq}, n_{12}^{eq}, n_{24}^{eq})$. It obviously satisfy the integer equation $n_1+4n_4+6n_6+12n_{12}+24n_{24} = \frac{(n-1)(n-2)(n-3)}{6}$. Numerical evidence shows that the minimum for this symmetry is smaller than the minima in other symmetry pattern cases. Therefore, we restrict ourselves to one more constraint:

c. The integer pair $(n_1, n_4, n_6, n_{12}, n_{24})$ for the symmetrical pattern of the degree n optimal set is $(n_1^{eq}, n_4^{eq}, n_6^{eq}, n_{12}^{eq}, n_{24}^{eq})$, the solution for the symmetrical pattern of the degree n equally spaced point set.

In Table 3, we list the approximate optimal sets in the tetrahedron barycentric coordinates: $b_1 = \frac{1}{2}(1 - x - \frac{y}{\sqrt{3}} - \frac{z}{\sqrt{6}})$, $b_2 = \frac{1}{2}(1 + x - \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{6}})$, $b_3 = \frac{1}{\sqrt{3}}(y - \frac{z}{\sqrt{6}})$, $b_4 = \sqrt{\frac{3}{8}}z$. $b_1 + b_2 + b_3 + b_4 = 1$. We also list $(n_1^{eq}, n_4^{eq}, n_6^{eq}, n_{12}^{eq}, n_{24}^{eq})$ and the number of interpolation points N_n. Points with multi-fold symmetry are listed only once. Other points with the same symmetry can be obtained by applying tetrahedron symmetries to the listed points, *i.e.*, by permuting barycentric coordinates b_1, b_2, b_3, b_4.

Table 2: Lebesgue constant and the mean norm for the optimal mean set, the VDM set and the equally spaced point set. Note the square of the mean norm is normalized by the tetrahedron volume.

n	$\lambda(T_2^n)$	$\lambda(T_{VDM}^n)$	$\lambda(T_{eq}^n)$	$\frac{3}{\sqrt{8}}\Vert\langle\mathcal{L}_{T_2^n}\rangle\Vert_2^2$	$\frac{3}{\sqrt{8}}\Vert\langle\mathcal{L}_{T_{VDM}^n}\rangle\Vert_2^2$	$\frac{3}{\sqrt{8}}\Vert\langle\mathcal{L}_{T_{eq}^n}\rangle\Vert_2^2$
2	2	2	2	$\frac{18}{35}$	$\frac{18}{35}$	$\frac{18}{35}$
3	2.9339	2.9329	3.0200	0.6435	0.6437	0.6893
4	4.1120	4.1534	4.8801	0.7669	0.7670	0.9329
5	5.6158	5.9961	8.0937	0.8892	0.8941	1.2915
6	7.3632	8.8898	13.6568	1.0237	1.0423	1.8721
7	9.3659	11.6425	23.3789	1.1895	1.2417	2.9157
8	12.3111	15.8340	40.5455	1.4165	1.5513	4.9854
9	15.6857	22.3304	71.1521	1.7532	2.0984	9.4538

Neither T_2^n nor T_{VDM}^n leads to the minimization of the Lebesgue constant. Nevertheless based on the one dimensional and the triangle results, we expect that the Lebesgue constant of T_2^n will not be significantly larger than the Lebesgue constant of T_1^n, and will be smaller than the Lebesgue constant of T_{VDM}^n. In Table 2, we give the Lebesgue constants for the sets T_2^n and T_{VDM}^n. For comparison, we also list the Lebesgue constant for the equally spaced point set $T_{eq}^n = \{(b_1 = \frac{i}{n}, b_2 = \frac{j}{n}, b_3 = \frac{k}{n}, b_4 = \frac{l}{n}), 0 \leq i,j,k,l; i+j+k+l = n\}$. Indeed, T_2^n has a smaller Lebesgue constant than T_{VDM}^n and a much smaller Lebesgue constant than that of T_{eq}^n. The last three columns give the mean norms.

We have given in this paper the mean optimal set T_2^n in the tetrahedron. T_2^n has the smallest Lebesgue constant known today. The other two interpolation sets are the equally spaced point set T_{eq}^n and the VDM set T_{VDM}^n.

Table 3: Barycentric coordinates for the tetrahedron optimal mean set T_2.

n	N_n	n_1	n_4	n_6	n_{12}	n_{24}	b_1	b_2	b_3	b_4
2	10	0	0	0	0	0	1.0000000	0.0000000	0.0000000	0.0000000
							0.5000000	0.5000000	0.0000000	0.0000000
3	20	0	0	0	0	0	1.0000000	0.0000000	0.0000000	0.0000000
							0.7251957	0.2748043	0.0000000	0.0000000
							0.3333333	0.3333333	0.3333333	0.0000000
4	35	1	0	0	0	0	1.0000000	0.0000000	0.0000000	0.0000000
							0.8306024	0.1693976	0.0000000	0.0000000
							0.5000000	0.5000000	0.0000000	0.0000000
							0.2208880	0.2208880	0.5582239	0.0000000
		1					0.2500000	0.2500000	0.2500000	0.2500000
5	56	0	1	0	0	0	1.0000000	0.0000000	0.0000000	0.0000000
							0.8866427	0.1133573	0.0000000	0.0000000
							0.6431761	0.3568239	0.0000000	0.0000000
							0.1525171	0.1525171	0.6949657	0.0000000
							0.4168658	0.4168658	0.1662683	0.0000000
			1				0.1823054	0.1823054	0.1823054	0.4530838
6	84	0	1	1	0	0	1.0000000	0.0000000	0.0000000	0.0000000
							0.9194021	0.0805979	0.0000000	0.0000000
							0.7349105	0.2650895	0.0000000	0.0000000
							0.5000000	0.5000000	0.0000000	0.0000000
							0.3333333	0.3333333	0.3333333	0.0000000
							0.1097139	0.1097139	0.7805723	0.0000000
							0.3157892	0.5586077	0.1256031	0.0000000
			1				0.1357838	0.1357838	0.1357838	0.5926485
				1			0.3559336	0.3559336	0.1440664	0.1440664
7	120	0	2	0	1	0	1.0000000	0.0000000	0.0000000	0.0000000
							0.9398927	0.0601073	0.0000000	0.0000000
							0.7957614	0.2042386	0.0000000	0.0000000
							0.6042138	0.3957862	0.0000000	0.0000000
							0.0817370	0.0817370	0.8365261	0.0000000
							0.4494208	0.4494208	0.1011584	0.0000000
							0.2663399	0.2663399	0.4673202	0.0000000
							0.2447528	0.6584392	0.0968080	0.0000000
			2				0.1046666	0.1046666	0.1046666	0.6860001
							0.2936310	0.2936310	0.2936310	0.1191069
					1		0.1141973	0.1141973	0.4885725	0.2830329

n	N_n	n_1	n_4	n_6	n_{12}	n_{24}	b_1	b_2	b_3	b_4
8	165	1	1	1	2	0	1.0000000	0.0000000	0.0000000	0.0000000
							0.9533797	0.0466203	0.0000000	0.0000000
							0.8375919	0.1624081	0.0000000	0.0000000
							0.6801403	0.3198597	0.0000000	0.0000000
							0.5000000	0.5000000	0.0000000	0.0000000
							0.0627331	0.0627331	0.8745338	0.0000000
							0.2153606	0.2153606	0.5692789	0.0000000
							0.3891297	0.3891297	0.2217406	0.0000000
							0.3657423	0.5524728	0.0817849	0.0000000
							0.1942206	0.7294168	0.0763626	0.0000000
		1					0.2500000	0.2500000	0.2500000	0.2500000
			1				0.0834511	0.0834511	0.0834511	0.7496468
				1			0.4060462	0.4060462	0.0939538	0.0939538
					2		0.0927818	0.0927818	0.5844475	0.2299889
							0.2432058	0.2432058	0.4157364	0.0978521
9	220	0	2	0	4	0	1.0000000	0.0000000	0.0000000	0.0000000
							0.9626819	0.0373181	0.0000000	0.0000000
							0.8672666	0.1327334	0.0000000	0.0000000
							0.7361751	0.2638249	0.0000000	0.0000000
							0.5815151	0.4184849	0.0000000	0.0000000
							0.3333333	0.3333333	0.3333333	0.0000000
							0.0493729	0.0493729	0.9012542	0.0000000
							0.4658361	0.4658361	0.0683277	0.0000000
							0.1769439	0.1769439	0.6461122	0.0000000
							0.3020146	0.6309227	0.0670627	0.0000000
							0.1575680	0.7808733	0.0615587	0.0000000
							0.3261032	0.4887991	0.1850977	0.0000000
			2				0.0682526	0.0682526	0.0682526	0.7952422
							0.2123036	0.2123036	0.2123036	0.3630891
					4		0.0774288	0.0774288	0.6543249	0.1908175
							0.0781245	0.0781245	0.5016150	0.3421360
							0.2049128	0.2049128	0.5083123	0.0818620
							0.3552639	0.3552639	0.2073089	0.0821632

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